

EQUILIBRIUM MEASURES AT TEMPERATURE ZERO FOR HÉNON-LIKE MAPS AT THE FIRST BIFURCATION

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ABSTRACT. We develop a thermodynamic formalism for a strongly dissipative Hénon-like map at the first bifurcation parameter at which the uniform hyperbolicity is destroyed by the formation of tangencies inside the limit set. For any $t \in \mathbb{R}$ we prove the existence of an invariant Borel probability measure which minimizes the free energy associated with a non continuous geometric potential $-t \log J^u$, where J^u denotes the Jacobian in the unstable direction. Under a mild condition, we show that any accumulation point of these measures as $t \rightarrow +\infty$ minimizes the unstable Lyapunov exponent. We also show that the equilibrium measures converge as $t \rightarrow -\infty$ to a Dirac measure which maximizes the unstable Lyapunov exponent.

1. INTRODUCTION

A basic problem in dynamics is to describe how structurally stable systems lose their stability through continuous modifications of the systems. The loss of stability of horseshoes through homoclinic bifurcations is modeled by a family of Hénon-like diffeomorphisms

$$(1) \quad f_a: (x, y) \in \mathbb{R}^2 \mapsto (1 - ax^2, 0) + b \cdot \Phi(a, b, x, y), \quad a \in \mathbb{R}, \quad 0 < b \ll 1.$$

Here, Φ is bounded continuous in (a, b, x, y) and C^2 in (a, x, y) . It is known [1, 7, 9, 22] that there is a *first bifurcation parameter* $a^* = a^*(b) \in \mathbb{R}$ with the following properties:

- $a^* \rightarrow 2$ as $b \rightarrow 0$;
- the non wandering set of f_a is a uniformly hyperbolic horseshoe for $a > a^*$;
- for $a = a^*$ there is a single orbit of homoclinic or heteroclinic tangency involving (one of) the two fixed saddles. The tangency is quadratic, and the family $\{f_a\}_{a \in \mathbb{R}}$ unfolds this tangency generically.

The study of the map f_{a^*} opens the door to understanding the dynamics beyond uniform hyperbolicity in dimension two. In this paper we advance the thermodynamic formalism for f_{a^*} initiated in [18, 19]. We prove the existence of equilibrium measures for a family $\{\varphi_t\}_{t \in \mathbb{R}}$ of non continuous geometric potentials, and study accumulation points of these measures as $t \rightarrow \pm\infty$.

Write f for f_{a^*} . The non wandering set of f , denoted by Ω , is a compact f -invariant set. Let $\mathcal{M}(f)$ denote the space of f -invariant Borel probability measures endowed with the topology of weak convergence. For a potential function $\varphi: \Omega \rightarrow \mathbb{R}$ the minus of the free energy $F_\varphi: \mathcal{M}(f) \rightarrow \mathbb{R}$ is defined by

$$F_\varphi(\mu) = h(\mu) + \int \varphi d\mu,$$

where $h(\mu)$ denotes the entropy of μ . An *equilibrium measure* for the potential φ is a measure $\mu_\varphi \in \mathcal{M}(f)$ which maximizes F_φ , i.e.,

$$F_\varphi(\mu_\varphi) = \sup\{F_\varphi(\mu) : \mu \in \mathcal{M}(f)\}.$$

The existence and uniqueness of equilibrium measures depend upon the characteristics of the system and the potential. The family of potentials we are concerned with is

$$\varphi_t = -t \log J^u \quad t \in \mathbb{R},$$

where J^u denotes the Jacobian in the *unstable direction* defined as follows. For a point $x \in \mathbb{R}^2$ let E_x^u denote the one-dimensional subspace of $T_x \mathbb{R}^2$ such that

$$(2) \quad \limsup_{n \rightarrow \infty} \frac{1}{n} \log \|D_x f^{-n}|E_x^u\| < 0.$$

Since f^{-1} expands area, the one-dimensional subspace of $T_x \mathbb{R}^2$ with this property is unique when it makes sense. We call E_x^u the *unstable direction at x* and define $J^u(x) = \|D_x f|E_x^u\|$. It was proved in [18, Proposition 4.1] that E_x^u makes sense for all $x \in \Omega$, and $x \in \Omega \mapsto E_x^u$ is continuous except at the fixed saddle near $(-1, 0)$ where it is merely measurable.

Since the chaotic behavior of f is created by the (non-uniform) expansion along the unstable direction, a good deal of information is obtained by studying the equilibrium measures for φ_t and the associated *pressure function* $t \in \mathbb{R} \mapsto P(t)$, where

$$P(t) = \sup\{F_{\varphi_t}(\mu) : \mu \in \mathcal{M}(f)\}.$$

The existence of equilibrium measures for φ_t was proved in [18] for all $t \leq 0$, and for those $t > 0$ such that $P(t)/t$ is slightly bigger than $-\log 2$. However, the arguments and the result in [18] do not cover sufficiently large $t > 0$. Our first theorem complements this point.

Theorem A. *Assume f preserves orientation. For any $t \in \mathbb{R}$ there exists an equilibrium measure for φ_t .*

For t in a large bounded interval, the uniqueness of equilibrium measures for φ_t was established in [19]. It would be nice to prove the uniqueness for all $t \in \mathbb{R}$, including the orientation reversing case.

Since t represents the inverse of the temperature in statistical mechanics, $t \rightarrow \pm\infty$ means that the temperature goes to zero. Hence, it is natural to study accumulation points of equilibrium measures for φ_t as $t \rightarrow \pm\infty$. They represent the lowest energy states, and may reflect the characteristics of the system.

The study of the behavior of the equilibrium measures as $t \rightarrow \pm\infty$ is also related to the ergodic optimization (See e.g. [3] and the references therein): given a continuous dynamical system T acting on a compact metric space X , and a real-valued function ϕ on X , one looks for T -invariant Borel probability measures which maximize the integral of ϕ . One way to do this is by freezing the system: to consider a family $\{t\phi\}_{t \in \mathbb{R}}$ of potentials and an associated family $\{\nu_t\}_{t \in \mathbb{R}}$ of equilibrium measures, and to let $t \rightarrow +\infty$. If the topological entropy is finite and the potential is continuous, then any accumulation point as $t \rightarrow +\infty$ maximizes the integral of ϕ . For uniformly hyperbolic systems or the subshift of finite type, the convergence has been established for certain locally constant potentials [5, 11] as well as for a residual set of continuous potentials [8, 10]. However, little is known for non hyperbolic systems.

An *unstable Lyapunov exponent* of a measure $\mu \in \mathcal{M}(f)$ is a number $\lambda^u(\mu)$ defined by

$$\lambda^u(\mu) = \int \log J^u d\mu.$$

Of interest to us are measures which optimize the unstable Lyapunov exponent. Since the unstable Lyapunov exponent is not continuous as a function of measures, the existence of such measures is an issue. We show that any accumulation point of the equilibrium measures for $\varphi_t = -t \log J^u$ as $t \rightarrow \pm\infty$ optimizes the unstable Lyapunov exponent.

Set

$$\lambda_m^u = \inf\{\lambda^u(\mu) : \mu \in \mathcal{M}(f)\}.$$

A measure $\mu \in \mathcal{M}(f)$ is called *Lyapunov minimizing* if $\lambda^u(\mu) = \lambda_m^u$. Let Q denote the fixed point of f near $(-1, 0)$, and δ_Q the Dirac measure at Q .

Theorem B. *Assume f preserves orientation. For $t \in \mathbb{R}$ let μ_t be an ergodic equilibrium measure for φ_t . Any accumulation point of $\{\mu_t\}_{t \in \mathbb{R}}$ as $t \rightarrow +\infty$ is δ_Q , or a Lyapunov minimizing measure. If $(1/2)\lambda^u(\delta_Q) \neq \lambda_m^u$, then any accumulation point of $\{\mu_t\}_{t \in \mathbb{R}}$ as $t \rightarrow +\infty$ is a Lyapunov minimizing measure.*

Since $\lambda^u(\delta_Q) \rightarrow \log 4$ and $\lambda_m^u \rightarrow \log 2$ as $b \rightarrow 0$, it is not easy to verify $(1/2)\lambda^u(\delta_Q) \neq \lambda_m^u$. However, from a given family (1) of Hénon-like diffeomorphisms one can construct another satisfying this condition by slightly perturbing Φ .

It is worthwhile to compare Theorem B with the results of Leplaideur [12]. In this paper, he studied an orientation preserving non-uniformly hyperbolic horseshoe map with three symbols, with a single orbit of homoclinic tangency, introduced in [17]. Although this map is similar to our f at a first glance, its equilibrium measures converge as $t \rightarrow +\infty$ to a Dirac measure which maximizes the unstable Lyapunov exponent. He also proved the nonexistence of a measure which minimizes the unstable Lyapunov exponent.

Since there may exist multiple Lyapunov minimizing measures of f , it is important to give a criterion for which one is “selected” in the limit $t \rightarrow +\infty$. The next theorem establishes a version of the “entropy criterion” in [3] for uniformly hyperbolic systems or the subshift of finite type with Hölder continuous potentials. Let us say that a Lyapunov minimizing measure $\mu \in \mathcal{M}(f)$ is *entropy maximizing* if

$$h(\mu) = \sup\{h(\nu) : \nu \in \mathcal{M}(f), \nu \text{ is Lyapunov minimizing}\}.$$

Theorem C. *Let f and $\{\mu_t\}_{t \in \mathbb{R}}$ be the same as in Theorem B. If $(1/2)\lambda^u(\delta_Q) \neq \lambda_m^u$, then any accumulation point of $\{\mu_t\}_{t \in \mathbb{R}}$ as $t \rightarrow +\infty$ is an entropy maximizing measure.*

We now turn to the case $t \rightarrow -\infty$. The next theorem holds regardless of the orientation of the map f .

Theorem D. *Let $\{\mu_t\}_{t \in \mathbb{R}}$ be such that μ_t is an ergodic equilibrium measure for φ_t for all $t \in \mathbb{R}$. Then μ_t converges to δ_Q as $t \rightarrow -\infty$.*

It follows from a proof of Theorem D that δ_Q is the unique measure which maximizes the unstable Lyapunov exponent (See Lemma 3.2). Apart from the uniqueness, the existence of such maximizing measures follows from the result in [6].

The rest of this paper consists of two sections. In Sect.2 we develop necessary tools, and prove the theorems in Sect.3. A main ingredient is a control of derivatives in the unstable direction. To recover from small derivatives near the point of tangency, we develop Benedicks

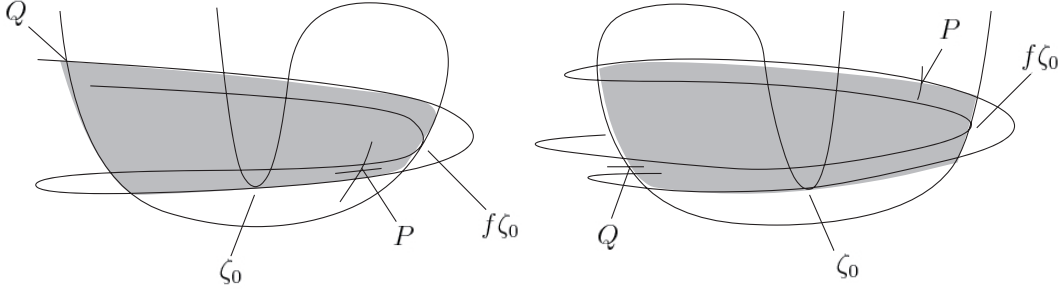


FIGURE 1. Manifold organization for $a = a^*$ in the case of the fold turning down: orientation preserving/reversing (left/right). The shaded domains represent the rectangle R containing the non wandering set Ω (see Sect.2.1).

& Carleson's critical point approach [2] further. The difference from the attractor case [2] is that all but one critical points escape to infinity under forward iteration. This issue has been successfully tackled in [18], but substantial improvements are necessary to treat all $t > 0$. In particular, both lower and upper estimates of derivatives are necessary, as stated in Proposition 2.2.

In Sect.2.5 we prove a key upper estimate of λ_m^u (See Corollary 2.12) needed for the proofs of Theorems A and B. Since each critical orbit spends most of its lifespan near the fixed saddle with a large derivative, the construction of measures with small unstable Lyapunov exponent involves a control of the position at which reference orbits are released from the effect of the critical orbits. We show that this is feasible for carefully chosen orbits, provided the map f preserves orientation.

2. PRELIMINARIES

In this section we develop necessary tools for the proofs of the theorems. For the rest of this paper we are concerned with the following constants: δ, b chosen in this order, the purposes of which are as follows:

- δ determines the size of a neighborhood of ζ_0 (See Sect.2.2);
- b determines the magnitude of the reminder term $b \cdot \Phi$ in (1).

We shall write C with or without indices to denote any constant which is independent of δ, b .

2.1. The non wandering set. The map f has exactly two fixed points, which are saddles. Let P denote the one near $(1/2, 0)$. Recall that Q is the other one near $(-1, 0)$. The orbit of tangency intersects a small neighborhood of the origin $(0, 0)$ exactly at one point, denoted by ζ_0 (FIGURE 1). If f preserves orientation, then $\zeta_0 \in W^s(Q) \cap W^u(Q)$. If f reverses orientation, then $\zeta_0 \in W^s(Q) \cap W^u(P)$.

If f preserves orientation, let $W^u = W^u(Q)$. Otherwise, let $W^u = W^u(P)$. By a *rectangle* we mean any compact domain bordered by two compact curves in W^u and two in the stable

manifolds of P or Q . By an *unstable side* of a rectangle we mean any of the two boundary curves in W^u . A *stable side* is defined similarly.

We define a rectangle containing the non wandering set. Let

$$V = \{(x, y) \in \mathbb{R}^2 : |x| < 2, |y| < \sqrt{b}\}.$$

By the results of [18] there exists a rectangle R in V with the following properties (See FIGURE 1):

- (R1) $\Omega = \{x \in R : f^n x \in R \text{ for every } n \in \mathbb{Z}\}$;
- (R2) one of the unstable sides of R contains ζ_0 ;
- (R3) one of the stable sides of R contains $f\zeta_0$. This side is denoted by α_0^+ . The other side, denoted by α_0^- , contains Q ;
- (R4) $f\alpha_0^+ \subset \alpha_0^-$.

2.2. Critical points. Set

$$I(\delta) = \{(x, y) \in V : |x| < \delta\}.$$

Observe that $\zeta_0 \in I(\delta)$, provided b is small enough. Although the dynamics outside of $I(\delta)$ is uniformly hyperbolic, returns to the inside of $I(\delta)$ are inevitable and must be treated with care. A key ingredient is the notion of critical points, i.e., points of tangencies between $C^2(b)$ -curves in W^u and preimages of leaves of a stable foliation. We quote results from [18] surrounding critical points, and shapen them further.

From the hyperbolicity of the saddle Q , there exist two mutually disjoint connected open sets U^- , U^+ independent of b such that $\alpha_0^- \subset U^-$, $\alpha_0^+ \subset U^+$, $U^+ \cap fU^+ = \emptyset = U^+ \cap fU^-$ and a foliation \mathcal{F}^s of $U = U^- \cup U^+$ by one-dimensional leaves such that:

- (F1) $\mathcal{F}^s(Q)$, the leaf of \mathcal{F}^s containing Q , contains α_0^- ;
- (F2) if $x, fx \in U$, then $f(\mathcal{F}^s(x)) \subset \mathcal{F}^s(fx)$;
- (F3) let $e^s(x)$ denote the unit vector in $T_x \mathcal{F}^s(x)$ whose second component is positive. Then $x \mapsto e^s(x)$ is C^1 , $\|D_x f e^s(x)\| \leq Cb$ and $\|D_x e^s(x)\| \leq C$;
- (F4) if $x, fx \in U$, then $s(e^s(x)) \geq C/\sqrt{b}$.

Here, a *slope* $s(v)$ of a nonzero tangent vector $v = \begin{pmatrix} \xi \\ \eta \end{pmatrix}$ at a point in \mathbb{R}^2 is defined by $s(v) = |\eta|/|\xi|$ if $\xi \neq 0$, and $s(v) = \infty$ if $\xi = 0$.

Definition 2.1. We say $\zeta \in W^u \cap I(\delta)$ is a *critical point* if $f\zeta \in U^+$ and $T_{f\zeta} W^u = T_{f\zeta} \mathcal{F}^s(f\zeta)$.

By a $C^2(b)$ -*curve* we mean a compact, nearly horizontal C^2 curve in V such that the slopes of tangent vectors to it are $\leq \sqrt{b}$ and the curvature is everywhere $\leq \sqrt{b}$. Let S denote the compact lenticular domain bounded by the parabola $f^{-1}\alpha_0^+ \cap R$ and the unstable side of R not containing ζ_0 . Let us record two properties of the critical points:

- (C1) any $C^2(b)$ -curve in $W^u \cap I(\delta)$ contains at most one critical point (See e.g. [21, Remark 2.4]);
- (C2) any critical point other than ζ_0 is contained in the interior of S . Hence it is mapped to the outside of R , and then escape to infinity under forward iteration.

(C2) implies that the critical orbits are contained in a region where the uniform hyperbolicity is apparent. Hence, by binding generic orbits which fall inside $I(\delta)$ to suitable critical points, and then copying the exponential growth along the critical orbits, one shows that the horizontal slopes and the expansion are restored after suffering from the loss due to the folding behavior near $I(\delta)$.

2.3. Binding argument. Let ζ be a critical point and $x \in I(\delta) \setminus S$. We say a unit tangent vector v at x is *in admissible position relative to ζ* if there exists a $C^2(b)$ -curve which is tangent to both $T_\zeta W^u$ and v .

Proposition 2.2. *Let ζ be a critical point, $x \in (\Omega \cap I(\delta)) \setminus S$ and v a unit tangent vector at x in admissible position relative to ζ . There exists a positive integer $p = p(\zeta, x)$ such that:*

- (a) $f^i \zeta, f^i x \in U$ for every $1 \leq i \leq p$;
- (b) $C_1 e^{\frac{p}{2} \lambda^u(\delta_Q)} \leq \|D_x f^p v\| \leq C_2 e^{\frac{p}{2} \lambda^u(\delta_Q)}$;
- (c) $s(D_x f^p v) \leq \sqrt{b}$;
- (d) if $\zeta = \zeta_0$, then $0 < C_3 \leq |f^p x - Q| \leq C_4 \ll 1$.

Proof. Let $\tau > 0$ be sufficiently small so that $\{y \in \mathbb{R}^2 : \min\{|y - z| : z \in \alpha_0^- \cup \alpha_0^+\} \leq \sqrt{\tau}\} \subset U$. For $i \geq 1$ write $w_i = D_{f\zeta} f^{i-1}(\frac{1}{0})$. For $k \geq 1$ define

$$D_k = \tau \left[\sum_{i=1}^k \frac{\|w_i\|^2}{\|w_{i+1}\|} \right]^{-1}.$$

Write $\mathcal{F}^s(f\zeta) = \{(F(y), y) : y \in J\}$, where J is an interval containing $[-\sqrt{b}, \sqrt{b}]$. Write $fx = (x_0, y_0)$, and let γ denote the segment connecting fx and $(F(y_0), y_0)$. Set $N = \sup\{i \geq 0 : f^i \zeta \in U\}$. We claim there exists a unique integer $p \in [1, N]$ such that

$$(3) \quad D_p < \text{length}(\gamma) \leq D_{p-1}.$$

Since $\zeta \neq x$ and $D_p \rightarrow 0$ as $p \rightarrow \infty$, this is obvious if $N = \infty$. In the case $N < \infty$, assume $|x_0 - F(y_0)| \leq D_N$. Then $|f^{N+1} \zeta - f^{N+1} x| \leq C\tau$. From the assumption $x \in \Omega$ and (R1), $f^{N+1} x \in R$. The definition of N gives $f^{N+1} \zeta \notin U$. Hence $|f^{N+1} \zeta - f^{N+1} x| \geq C\sqrt{\tau}$ and we obtain a contradiction for sufficiently small τ . So the claim holds.

For $A, B > 0$ we write $A \approx B$ if both A/B and B/A are bounded from above by constants independent of τ, δ, b .

Lemma 2.3. *For every $k \leq N$,*

- (a) $D_k \approx \tau e^{-\lambda^u(\delta_Q)k}$;
- (b) $\|w_k\| D_k \approx \tau$.

Proof. By the bounded distortion results in [15, Section 6] and [18, Lemma 2.6(a)], $\|w_i\| \approx e^{\lambda^u(\delta_Q)(i-1)}$ holds for every $1 \leq i \leq k+1$. Hence

$$D_k^{-1} = \frac{1}{\tau} \sum_{i=1}^k \frac{\|w_i\|^2}{\|w_{i+1}\|} \approx \frac{1}{\tau} \sum_{i=1}^k \|w_k\| e^{\lambda^u(\delta_Q)(i-k)} \approx \frac{1}{\tau} e^{\lambda^u(\delta_Q)k},$$

and so (a) holds. (b) is contained in [18, Lemma 2.4]. \square

If $1 \leq i \leq p-1$, then by [18, Lemma 2.6], $\|D_z f^i(\frac{1}{0})\| \approx \|w_{i+1}\|$ holds for all $z \in \gamma$, and $f^i \gamma$ is a $C^2(b)$ -curve. Lemma 2.3(b) gives

$$\text{length}(f^i \gamma) \approx \text{length}(\gamma) \|w_{i+1}\| \leq C D_{p-1} \|w_{i+1}\| \leq C D_{p-1} \|w_p\| \leq C\tau.$$

This implies $x, fx, \dots, f^p x \in U$, and so (a).

Split

$$D_x f v = A \cdot \left(\frac{1}{0}\right) + B \cdot e^s(fx), \quad A, B \in \mathbb{R}.$$

Since v is in admissible position relative to ζ , from the results in [18, 20] we have $A \approx |\zeta - x|$ and $\text{length}(\gamma) \approx |\zeta - x|^2$. On the other hand, (3) and Lemma 2.3(a) give $\text{length}(\gamma) \approx D_p \approx \tau e^{-\lambda^u(\delta_Q)p}$. Hence

$$|\zeta - x| \approx \frac{1}{\sqrt{\tau}} e^{-\frac{p}{2}\lambda^u(\delta_Q)}.$$

By Lemma 2.3(b),

$$(4) \quad \text{length}(f^{p-1}\gamma) \approx D_p \|w_p\| \approx \tau.$$

Putting these estimates together we obtain

$$\begin{aligned} |A| \cdot \|D_{fx} f^{p-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix}\| &\approx |\zeta - x| \cdot \|w_p\| \approx |\zeta - x| \cdot \frac{\text{length}(f^{p-1}\gamma)}{\text{length}(\gamma)} \\ &\approx \text{length}(f^{p-1}\gamma) \cdot |\zeta - x|^{-1} \approx \tau^{\frac{3}{2}} e^{\frac{p}{2}\lambda^u(\delta_Q)}. \end{aligned}$$

For the other component in the splitting, (F3) gives

$$|B| \cdot \|D_{fx} f^{p-1} e^s(fx)\| \leq (Cb)^{p-1}.$$

Then $\|D_x f^p v\| \approx \tau^{\frac{3}{2}} e^{\frac{p}{2}\lambda^u(\delta_Q)}$, and so (b) holds. It also follows that $s(D_x f^{p-1}v) \ll 1$, and so $s(D_x f^p v) \leq \sqrt{b}$, and (c). (d) follows from (4) and $\mathcal{F}^s(f\zeta_0) \supset \alpha_0^+$, which is a consequence of (F1) (F2) and (R4). \square

Remark 2.4. The existence of a uniform lower bound on $|f^p x - f^p \zeta|$ can be read out from the above proof. However, this does not imply a uniform lower bound on $|f^p x - Q|$ as in Proposition 2.2(d), because if $\zeta \neq \zeta_0$ then $f^p \zeta$ escapes from R to the left of α_0^- .

The integer p and ζ in Proposition 2.2 are called a *bound period*, and a *binding critical point* of x respectively. The next lemma allows us to find a binding critical point for any non wandering point which falls inside $I(\delta)$. For $x \in \Omega$ let $e^u(x)$ denote any unit tangent vector which spans E_x^u .

Lemma 2.5. [18, Lemma 2.9] *For any $x \in \Omega \cap I(\delta) \setminus \{\zeta_0\}$ there exists a critical point relative to which $e^u(x)$ is in admissible position.*

2.4. Unstable Lyapunov exponents of limit measures. The next proposition, which is a substantial improvement of [18, Proposition 4.3], gives a lower estimate of the amount of drop of the unstable Lyapunov exponent in the weak limit of measures. Let $\mathcal{M}^e(f)$ denote the set of elements of $\mathcal{M}(f)$ which are ergodic.

Proposition 2.6. *Let $\{\mu_n\}_n$ be a sequence in $\mathcal{M}^e(f)$ such that $\mu_n \rightarrow \mu$, $\mu = u\delta_Q + (1-u)\nu$, $0 \leq u \leq 1$, $\nu \in \mathcal{M}(f)$ and $\nu\{Q\} = 0$. Then:*

$$(5) \quad \frac{u}{2} \lambda^u(\delta_Q) + (1-u) \lambda^u(\nu) \leq \liminf_{n \rightarrow \infty} \lambda^u(\mu_n);$$

$$(6) \quad \limsup_{n \rightarrow \infty} \lambda^u(\mu_n) \leq \lambda^u(\mu).$$

Proof. (6) was proved in the proof of [18, Proposition 4.3]. Here we prove (5).

Lemma 2.7. ([18, Lemma 4.4]) *Let $\{\mu_n\}_n$ be a sequence in $\mathcal{M}^e(f)$ such that $\mu_n \rightarrow \mu$ and $\mu\{Q\} = 0$. Then $\lambda^u(\mu_n) \rightarrow \lambda^u(\mu)$.*

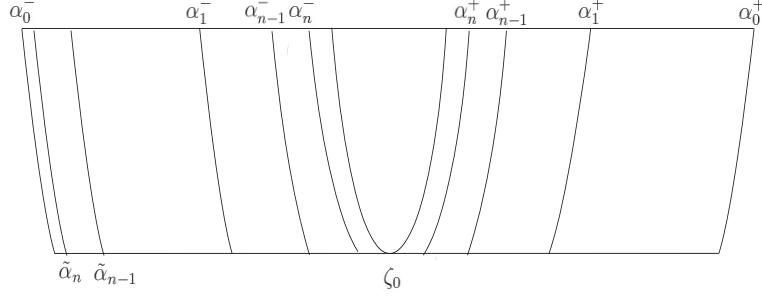


FIGURE 2. The curves $\{\tilde{\alpha}_n\}$, $\{\alpha_n^+\}$, $\{\alpha_n^-\}$. The $\{\tilde{\alpha}_n\}$ accumulate on the left stable side of R . Both $\{\alpha_n^+\}$ and $\{\alpha_n^-\}$ accumulate on the parabola $f^{-1}\alpha_0^+ \cap R$ containing the point of tangency ζ_0 near the origin.

In the case $u = 0$, (5) is a consequence of Lemma 2.7. We now consider the case $u \neq 0$. We begin by introducing a sequence $\{\tilde{\alpha}_k\}_{k=0}^\infty$ of compact curves in $W^s(P) \cap R$ which allow us to relate the proximity of an orbit's return to the boundary of the region S with the time it will subsequently spend near Q . Define $\tilde{\alpha}_0$ to be the connected component of $W^s(P) \cap R$ containing P . Given $\tilde{\alpha}_{k-1}$, define $\tilde{\alpha}_k$ to be one of the two components of $f^{-1}\tilde{\alpha}_{k-1} \cap R$ which is at the left of ζ_0 (See FIGURE 2).

Let $c \in (0, 1/2)$ and define

$$X(c) = \frac{\lambda^u(\delta_Q)}{5} \left(\frac{1}{2} - c \right) \in (0, 1).$$

Let \tilde{V}_k denote the rectangle bordered by $\tilde{\alpha}_k$, α_0^- and the unstable sides of R . Define

$$V_{c,k} = \bigcup_{i=0}^{[1-X(c)]k} f^i \tilde{V}_k,$$

where $[\cdot]$ denotes the integer part. Observe that $\{V_{c,k}\}_k$ is decreasing in k . By the Inclination Lemma, the Hausdorff distance between $\tilde{\alpha}_k$ and α_0^- converges to 0 as $k \rightarrow \infty$. This implies $\bigcap_{k=1}^\infty V_{c,k} = \alpha_0^-$.

Lemma 2.8. *If $0 < c_0 < c < 1/2$, then there exists $k_0 \geq 1$ such that if $k \geq k_0$ and $x \in \Omega$, $n \geq 1$ are such that $f^{-2}x \in I(\delta)$, $x \in \tilde{V}_k$, $x, fx, \dots, f^{n-1}x \in V_{c,k}$ and $f^n x \notin V_{c,k}$, then*

$$\|D_x f^n | E_x^u\| \geq e^{c_0 \lambda^u(\delta_Q)n}.$$

Proof. Write y for $f^{-2}x$. By Lemma 2.5 there exists a critical point ζ relative to which $e^u(y)$ is in admissible position. Let $p = p(\zeta, y)$ denote the corresponding bound period. We treat two cases separately.

Case 1: $p-2 \leq n$. Proposition 2.2 gives $\|D_y f^p e^u(y)\| \geq C e^{\frac{\lambda^u(\delta_Q)}{2}p}$ and $s(e^u(f^p y)) \leq \sqrt{b}$. Since $f^p y, f^{p+1}y, \dots, f^{n+1}y$ are located near Q , the bounded distortion gives $\|D_{f^p y} f^{n+2-p} e^u(f^p y)\| \geq$

$Ce^{\lambda^u(\delta_Q)(n+2-p)}$. Then

$$\begin{aligned}\|D_x f^n e^u(x)\| &= \frac{\|D_y f^{n+2} e^u(y)\|}{\|D_y f^2 e^u(y)\|} > \|D_y f^{n+2} e^u(y)\| \\ &= \|D_{f^p y} f^{n+2-p} e^u(f^p y)\| \cdot \|D_y f^p e^u(y)\| \\ &\geq C e^{\frac{\lambda^u(\delta_Q)}{2}(n+2)} \geq e^{c_0 \lambda^u(\delta_Q)n},\end{aligned}$$

where the last inequality holds for sufficiently large k because of $n > k$.

Case 2: $p - 2 > n$. Fix a $C^2(b)$ -curve γ which connects $f^{-1}x$ and α_0^+ . The curves $f^i \gamma$ ($i = 1, \dots, n+1$) are $C^2(b)$ -curves located near Q . The condition $f^{n-1}x \in V_k$ and $f^n x \notin V_k$ implies

$$(7) \quad \text{length}(f^{n+1}\gamma) > 5^{-X(c)k}.$$

The bounded distortion gives $\|D_z f^{n+1}(\frac{1}{0})\| \approx \|D_{f\zeta} f^{n+1}(\frac{1}{0})\|$ for all $z \in \gamma$. Using this and $\text{length}(\gamma) \leq C|\zeta - y|^2$ we have

$$(8) \quad \text{length}(f^{n+1}\gamma) \approx \text{length}(\gamma) \|D_{f\zeta} f^{n+1}(\frac{1}{0})\| \leq C|\zeta - y|^2 \|D_{f\zeta} f^{n+1}(\frac{1}{0})\|.$$

Proposition 2.3(a) gives

$$(9) \quad |\zeta - y| \leq C e^{-\frac{p}{2}\lambda^u(\delta_Q)}.$$

Split

$$D_y f e^u(y) = A \cdot (\frac{1}{0}) + B \cdot e^s(fy), \quad A, B \in \mathbb{R}.$$

Using (7) (8) (9) and $k < p$, for some $C > 0$ we have

$$\begin{aligned}|A| \cdot \|D_{fy} f^{n+1}(\frac{1}{0})\| &\approx |\zeta - y| \cdot \|D_{f\zeta} f^{n+1}(\frac{1}{0})\| \\ &\geq C \cdot \text{length}(f^{n+1}\gamma) |\zeta - y|^{-1} \geq C \cdot 5^{-X(c)p} e^{\frac{p}{2}\lambda^u(\delta_Q)} \\ &\geq C \cdot 2^{X(c)p} e^{c\lambda^u(\delta_Q)p} \geq e^{c\lambda^u(\delta_Q)p},\end{aligned}$$

where the last inequality holds provided k is sufficiently large so that $C \cdot 2^{X(c)k} \geq 1$.

For the other component in the splitting we have

$$|B| \cdot \|D_{fy} f^{n+1} e^s(fy)\| \leq (Cb)^{n+1} \leq (Cb)^p.$$

We have

$$\|D_x f^n e^u(x)\| > \|D_y f^{n+2} e^u(y)\| \geq e^{c\lambda^u(\delta_Q)p} - (Cb)^p \geq e^{c_0 \lambda^u(\delta_Q)p} > e^{c_0 \lambda^u(\delta_Q)n}.$$

where the second last inequality holds sufficiently large k because of $c > c_0$ and $p > n + 2 > k$. \square

Let us return to the proof of (5) in the case $u \neq 0$. Taking a subsequence if necessary, we may assume $\{\lambda^u(\mu_n)\}_n$ converges. Let $0 < c_0 < c < 1/2$. Fix a partition of unity $\{\rho_{0,c,k}, \rho_{1,c,k}\}$ on R such that

$$\text{supp}(\rho_{0,c,k}) = \overline{\{x \in R: \rho_{0,c,k}(x) \neq 0\}} \subset V_{c,k} \quad \text{and} \quad \text{supp}(\rho_{1,c,k}) \subset R \setminus V_{c,2k}.$$

Claim 2.9. $\liminf_{n \rightarrow \infty} \mu_n(V_{c,k}) \geq u$.

Proof. If $u \neq 1$, then $\mu_n - u\delta_Q \rightarrow (1-u)\nu$. Since $\nu\{Q\} = 0$, $\nu(\partial V_{c,k}) = 0$. This yields $(\mu_n - u\delta_Q)(V_{c,k}) \rightarrow (1-u)\nu(V_{c,k})$ as $n \rightarrow \infty$, namely $\mu_n(V_{c,k}) \rightarrow u + (1-u)\nu(V_{c,k})$. The same convergence obviously takes place in the case $u = 1$. Hence the claim holds. \square

From the Ergodic Theorem, there exists $\xi_n \in \Omega$ such that

$$\lim_{m \rightarrow \infty} \frac{1}{m} \# \{0 \leq i \leq m-1 : f^i \xi_n \in V_{c,k}\} = \mu_n(V_{c,k}).$$

The forward orbit of ξ_n is a concatenation of segments in $V_{c,k}$ and those out of $V_{c,k}$. Lemma 2.8 gives

$$\int \rho_{0,c,k} \log J^u d\mu_n = \lim_{m \rightarrow \infty} \frac{1}{m} \sum_{i=0}^{m-1} (\rho_{0,c,k} \log J^u) \circ f^i(\xi_n) \geq \mu_n(V_{c,k}) c_0 \lambda^u(\delta_Q).$$

If $u \neq 1$, then the weak convergence for the sequence $\{(1-u)^{-1}(\mu_n - u\delta_Q)\}_n$ in $\mathcal{M}(f)$ implies

$$\lim_{n \rightarrow \infty} \int \rho_{1,c,k} \log J^u d\mu_n = (1-u) \int \rho_{1,c,k} \log J^u d\nu.$$

The same inequality remains to hold for the case $u = 1$. Hence we have

$$\begin{aligned} \lim_{n \rightarrow \infty} \lambda^u(\mu_n) &= \lim_{n \rightarrow \infty} \int \rho_{0,c,k} \log J^u d\mu_n + \lim_{n \rightarrow \infty} \int \rho_{1,c,k} \log J^u d\mu_n \\ &\geq u c_0 \lambda^u(\delta_Q) + (1-u) \int \rho_{1,c,k} \log J^u d\nu. \end{aligned}$$

Since $\nu\{Q\} = 0$, $\rho_{1,c,k} \log J^u \rightarrow \log J^u$ ν -a.e. as $k \rightarrow \infty$. Letting $k \rightarrow \infty$ and then using the Dominated Convergence Theorem gives

$$\lim_{n \rightarrow \infty} \lambda^u(\mu_n) \geq u c_0 \lambda^u(\delta_Q) + (1-u) \lambda^u(\nu).$$

Since c, c_0 are arbitrary such that $0 < c_0 < c < 1/2$, the desired inequality holds. \square

2.5. Construction of measures with small unstable Lyapunov exponents. In this subsection we construct a sequence of atomic measures with small unstable Lyapunov exponent. In this and the next subsections we assume f preserves orientation.

We continue using the sequence $\{\tilde{\alpha}_n\}_{n=0}^\infty$ of compact curves in $W^s(P) \cap R$ in the proof of Proposition 2.6. For each $n \geq 0$ let α'_n denote the connected component of $f^{-1}\tilde{\alpha}_n \cap R$ which is not $\tilde{\alpha}_{n+1}$. The set $f^{-1}\alpha'_n \cap R$ consists of two curves, one at the left of ζ_0 and the other at the right. They are denoted by α_{n+1}^- , α_{n+1}^+ respectively (See FIGURE 2). By definition, these curves obey the following diagram

$$(10) \quad \{\alpha_{n+1}^-, \alpha_{n+1}^+\} \xrightarrow{f^2} \tilde{\alpha}_n \xrightarrow{f} \tilde{\alpha}_{n-1} \xrightarrow{f} \tilde{\alpha}_{n-2} \xrightarrow{f} \cdots \xrightarrow{f} \tilde{\alpha}_1 = \alpha_1^- \xrightarrow{f} \tilde{\alpha}_0 = \alpha_1^+.$$

Observe that $\tilde{\alpha}_0 = \alpha_1^+$ and $\tilde{\alpha}_1 = \alpha_1^-$.

Let ω_n^+ (resp. ω_n^-) denote the rectangle bordered by α_n^+ , α_{n+1}^+ (resp. α_n^- , α_{n+1}^-) and the unstable sides of R . The following holds:

- $f\omega_n^\pm$ is at the right of α_1^\pm . If $n \geq 2$, then $f^i\omega_n^\pm$ is at the left of α_1^\pm for every $2 \leq i \leq n$;
- One of the stable sides of $f^{n+1}\omega_n^\pm$ is contained in α_1^- and the other in α_1^+ . The unstable sides of $f^{n+1}\omega_n^\pm$ are $C^2(b)$ -curves connecting α_1^- and α_1^+ [19].

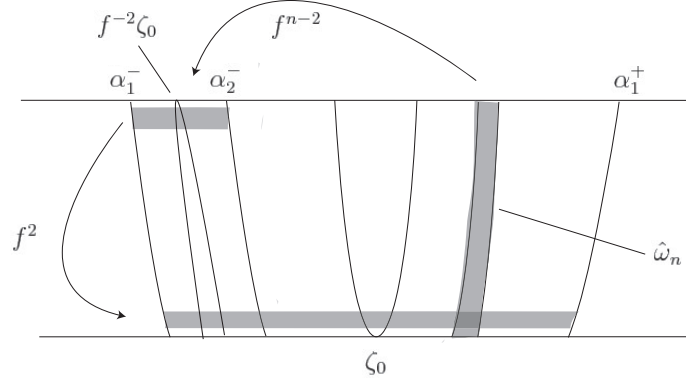
For each $n \geq 4$ define

$$\hat{\omega}_n = \omega_{n-3}^+ \cap f^{-n+2}\omega_1^-.$$

Since f preserves orientation, $f^{-2}\zeta_0$ is contained in the unstable sides of ω_1^- (See FIGURE 3).

Let $\gamma^u(\zeta_0)$ denote the $C^2(b)$ -curve in W^u which contains ζ_0 , and connects α_1^- and α_1^+ . Define

$$A_n = \{x \in \hat{\omega}_n \cap \Omega : \min\{|x - y| : y \in \gamma^u(\zeta_0)\} \leq (Cb)^{\frac{n}{2}}\}.$$

FIGURE 3. The rectangles $\hat{\omega}_n$, $f^{n-2}\hat{\omega}_n$, $f^n\hat{\omega}_n$.

If $x \in A_n$ then $|x - \zeta_0| > 5^{-n}$, for otherwise $f^n x$ were close to $f^n \zeta_0 \in \alpha_0^-$, a contradiction.

Proposition 2.10. *Assume f preserves orientation. For sufficiently large n and all $x \in A_n$,*

$$\|D_x f^n|E_x^u\| \leq C e^{\frac{n}{2}\lambda^u(\delta_Q)}.$$

Proof. Let n be large enough so that $A_n \subset I(\delta)$. Let $x \in A_n$. We show that $e^u(x)$ is in admissible position relative to ζ_0 . Since $\gamma^u(\zeta_0)$ is $C^2(b)$ and $|x - \zeta_0| > 5^{-n}$, this would hold if $|x - z| \leq C b^{\frac{n}{2}}$ and $\angle(E_x^u, T_z \gamma^u(\zeta_0)) \leq C b^{\frac{n}{8}}$, where z denote the point on $\gamma^u(\zeta_0)$ whose first coordinate coincides with that of x . The first inequality immediately follows from the definition of A_n . The second one follows from the sublemma below, combined with the following fact from [18, Sect.2.4 & Lemma 2.8]: there are a sequence $\{x_n\}_n$ in Ω and a sequence $\{\gamma_n\}_n$ of $C^2(b)$ -curves in W^u connecting α_1^- and α_1^+ such that $x_n \in \gamma_n$ and $T_{x_n} \gamma_n \rightarrow E_x^u$ as $n \rightarrow \infty$.

Sublemma 2.11. *Let $L \in (0, 1/4)$, and let $\gamma_i = \{(x, \gamma_i(x)) \in \mathbb{R}^2 : x \in [-L, L]\}$, $i = 1, 2$ be two disjoint $C^2(b)$ -curves. Assume $|\gamma_1(0) - \gamma_2(0)| \leq L^2$. Then $|\gamma_1'(0) - \gamma_2'(0)| \leq \sqrt{L}$.*

Proof. Without loss of generality we may assume γ_2 lies above γ_1 , and $\gamma_1'(0) > \gamma_2'(0)$. Set $A(x) = \gamma_2(x) - \gamma_1(x)$. By the assumption, $A(x) > 0$ for all $x \in [-L, L]$ and $A'(0) < 0$. By the $C^2(b)$ -property, $|A''(x)| \leq 2\sqrt{b}$. If $A'(0) < -\sqrt{L}$, then by the Mean Value Theorem, $A'(x) \leq A'(0) + 2\sqrt{b}x \leq -\sqrt{L}/2$, and thus $A(L) = A(0) + \int_0^L A'(x)dx \leq L^2 - L^{3/2}/2 < 0$, a contradiction. \square

Let $p = p(\zeta_0, x)$ denote the bound period. Proposition 2.2(d) implies $n - p \leq C$, and therefore

$$\|D_x f^n|E_x^u\| = \|D_x f^p|E_x^u\| \cdot \|D_{f^p x} f^{n-p}|E_{f^p x}^u\| \leq C e^{\frac{p}{2}\lambda^u(\delta_Q)} \cdot 5^{n-p} \leq C e^{\frac{n}{2}\lambda^u(\delta_Q)}. \quad \square$$

Corollary 2.12. *If f preserves orientation, then $(1/2)\lambda^u(\delta_Q) \geq \lambda_m^u$.*

Proof. By the result in [23], the subset $\bigcap_{k=-\infty}^{\infty} (f^n)^k \hat{\omega}_n$ of A_n is a singleton which consists of a hyperbolic periodic point of period n . Proposition 2.10 gives an upper estimate of the unstable Lyapunov exponent of the atomic probability measure on the orbit of this periodic point. Since n is arbitrary, this yields the desired inequality. \square

2.6. Lower estimate of the pressure. The next lemma will be used to derive a contradiction in the proof of Theorem A.

Lemma 2.13. *Assume f preserves orientation. If $(1/2)\lambda^u(\delta_Q) = \lambda_m^u$, then $P(t) > -t\lambda_m^u$ for any $t > 0$.*

Proof. We adapt the construction of Leplaideur [12] which was inspired by Makarov and Smirnov [16]. The idea is to construct a uniformly hyperbolic subset which supports an invariant measure whose minus of the free energy is slightly bigger than $-t\lambda_m^u$.

Let $q > 0$ be a square of a large integer. We use the rectangles $\hat{\omega}_n$ ($n = q - \sqrt{q} + 1, \dots, q$) to construct an induced system. Set $r_n = q - \sqrt{q} + n$ ($n = 1, \dots, \sqrt{q}$). Endow $\Sigma_{\sqrt{q}} = \{\underline{a} = \{a_i\}_{i \in \mathbb{Z}} : a_i \in \{1, \dots, \sqrt{q}\}\}$ with the product topology of the discrete topology. Define $\pi : \Sigma_{\sqrt{q}} \rightarrow \Omega$ by $\pi(\underline{a}) = x$, where

$$\omega_k^s = \hat{\omega}_{a_0} \cap \left(\bigcap_{i=1}^k f^{-r_{a_0}} \circ \dots \circ f^{-r_{a_{i-1}}} \hat{\omega}_{a_i} \right) \quad \text{and} \quad \omega_k^u = \bigcap_{i=1}^k f^{r_{a_{-1}}} \circ \dots \circ f^{r_{a_{-i}}} \hat{\omega}_{a_{-i}},$$

and

$$\{x\} = \left(\bigcap_{k=1}^{\infty} \omega_k^s \right) \cap \left(\bigcap_{k=1}^{\infty} \omega_k^u \right).$$

By [23], π is well-defined, continuous, injective.

Let $\sigma : \Sigma_{\sqrt{q}} \odot$ denote the left shift. For a σ^q -invariant Borel probability measure μ , define a measure $\mathcal{L}(\mu)$ by

$$\mathcal{L}(\mu) = \sum_{[a_0, a_1, \dots, a_{q-1}]} \sum_{i=0}^{r_{a_0} + r_{a_1} + \dots + r_{a_{q-1}} - 1} f_*^i(\pi_*(\mu|_{[a_0, a_1, \dots, a_{q-1}]})),$$

where $[a_0, a_1, \dots, a_{q-1}] = \{\underline{b} \in \Sigma_{\sqrt{q}} : b_i = a_i \ i = 0, 1, \dots, q-1\}$. Then $\mathcal{L}(\mu)$ is a probability and f -invariant. Define $r : \Sigma_{\sqrt{q}} \rightarrow \mathbb{R}$ by $r(\underline{a}) = \sum_{i=0}^{q-1} r_{a_i}$, and $\Phi_t : \Sigma_{\sqrt{q}} \rightarrow \mathbb{R}$ by

$$\Phi_t(\underline{a}) = \sum_{i=0}^{r(\underline{a})-1} \varphi_t(f^i(\pi(\underline{a}))).$$

Set $P_n = \{\underline{a} \in \Sigma_{\sqrt{q}} : \sigma^{qn}(\underline{a}) = \underline{a}\}$. By Proposition 2.10 and the assumption $(1/2)\lambda^u(\delta_Q) = \lambda_m^u$, for each $\underline{a} \in P_n$ we have

$$\frac{\Phi_t(\underline{a})}{r(\underline{a})} \geq -\frac{r_{a_0}Ct + (r_{a_1} + \dots + r_{a_{q-1}} - 1)t\lambda_m^u}{r_{a_1} + \dots + r_{a_{q-1}} - 1} \geq -\frac{Ct}{q} - t\lambda_m^u.$$

Let μ_0 denote the measure of maximal entropy of σ^q . Since r and Φ_t are continuous, as $n \rightarrow \infty$ we have

$$\frac{1}{\#P_n} \sum_{\underline{a} \in P_n} r(\underline{a}) \rightarrow \int r d\mu_0 \quad \text{and} \quad \frac{1}{\#P_n} \sum_{\underline{a} \in P_n} \Phi_t(\underline{a}) \rightarrow \int \Phi_t d\mu_0.$$

Hence

$$\frac{\int \Phi_t d\mu_0}{\int r d\mu_0} \geq -\frac{Ct}{q} - t\lambda_m^u.$$

Since the entropy of μ_0 is $q \log \sqrt{q}$ and $\int r d\mu_0 \leq q^2$, we obtain

$$\begin{aligned} h(\mathcal{L}(\mu_0)) - t \int \log J^u d\mathcal{L}(\mu_0) &= \frac{1}{\int r d\mu_0} \left(q \log \sqrt{q} + \int \Phi_t d\mu_0 \right) \\ &\geq \frac{1}{q} \log \sqrt{q} - t \frac{C}{q} - t \lambda_m^u > -t \lambda_m^u. \end{aligned}$$

The last inequality holds for sufficiently large q . \square

3. PROOFS OF THE THEOREMS

In this section we put together the results in Sect.2 and prove the theorems.

3.1. Existence of equilibrium measures. We prove Theorem A.

Proof of Theorem A. Let $t > 0$. Corollary 2.12 gives

$$(11) \quad P(t) \geq -t \lambda_m^u \geq -(t/2) \lambda^u(\delta_Q).$$

By the linearity of entropy and unstable Lyapunov exponent on measures, one can choose a sequence $\{\mu_n\}_n$ in $\mathcal{M}^e(f)$ such that $F_{\varphi_t}(\mu_n) \rightarrow P(t)$. Choosing a subsequence if necessary we may assume $\mu_n \rightarrow \mu \in \mathcal{M}(f)$. Write $\mu = u\delta_Q + (1-u)\nu$, $0 \leq u \leq 1$, $\nu\{Q\} = 0$. From the upper semi-continuity of entropy [18, Corollary 3.2] and (5),

$$\begin{aligned} (12) \quad P(t) &= \lim_{n \rightarrow \infty} F_{\varphi_t}(\mu_n) \leq \limsup_{n \rightarrow \infty} h(\mu_n) - t \liminf_{n \rightarrow \infty} \lambda^u(\mu_n) \\ &\leq h(\mu) - t \left(\frac{u}{2} \lambda^u(\delta_Q) + (1-u) \lambda^u(\nu) \right) \\ &= (1-u) F_{\varphi_t}(\nu) - \frac{tu}{2} \lambda^u(\delta_Q). \end{aligned}$$

For the last equality we have used $h(\mu) = (1-u)h(\nu)$. Plugging $-(tu/2)\lambda^u(\delta_Q) \leq uP(t)$ from (11) into (12) we obtain

$$(13) \quad P(t) \leq (1-u) F_{\varphi_t}(\nu) + uP(t).$$

If $u \neq 1$, then rearranging (13) yields $P(t) \leq F_{\varphi_t}(\nu)$. Namely ν is an equilibrium measure for φ_t . If $u = 1$, then (12) and Corollary 2.12 yield $P(t) \leq -(t/2)\lambda^u(\delta_Q) \leq -t\lambda_m^u$. From this and (11) we obtain $P(t) = -t\lambda_m^u = -(t/2)\lambda^u(\delta_Q)$, a contradiction to Lemma 2.13. \square

3.2. Accumulation points of equilibrium measures as $t \rightarrow +\infty$. We now prove Theorem B.

Proof of Theorem B. Let $\{\mu_{t_n}\}_{n=0}^\infty$ be a sequence in $\mathcal{M}(f)$ such that $t_n \nearrow +\infty$, μ_{t_n} is an ergodic equilibrium measure for φ_{t_n} and $\mu_{t_n} \rightarrow \mu$ as $n \rightarrow \infty$. Write $\mu = u\delta_Q + (1-u)\nu$, $0 \leq u \leq 1$, $\nu\{Q\} = 0$. Proposition 2.6 gives

$$(14) \quad \lambda_m^u = \lim_{n \rightarrow \infty} \lambda^u(\mu_{t_n}) \geq \frac{u}{2} \lambda^u(\delta_Q) + (1-u) \lambda^u(\nu).$$

By Corollary 2.12, $(1/2)\lambda^u(\delta_Q) \geq \lambda_m^u$. In this paragraph we treat the case $(1/2)\lambda^u(\delta_Q) > \lambda_m^u$. If $u = 1$, then (14) gives $\lambda_m^u \geq (1/2)\lambda^u(\delta_Q)$, a contradiction. Hence $u \neq 1$. Plugging $(u/2)\lambda^u(\delta_Q) \geq u\lambda_m^u$ into (14) and then rearranging the result yield $(1-u)\lambda_m^u \geq (1-u)\lambda^u(\nu)$, and thus $\lambda_m^u \geq \lambda^u(\nu)$. If $u \neq 0$, all these three inequalities are strict and we reach a contradiction. Hence $u = 0$ and so $\mu = \nu$. Lemma 2.7 gives $\lambda^u(\mu) = \lim_{n \rightarrow \infty} \lambda^u(\mu_{t_n}) = \lambda_m^u$.

In the case $(1/2)\lambda^u(\delta_Q) = \lambda_m^u$, (14) gives $(1-u)\lambda_m^u \geq (1-u)\lambda^u(\nu)$. If $u \neq 1$, then ν is Lyapunov minimizing. If $u = 1$, then $\mu = \delta_Q$. \square

3.3. Entropy criterion. We now prove Theorem C.

Proof of Theorem C. Let $\{\mu_{t_n}\}_{n=0}^\infty$ and μ be the same as in the proof of Theorem B. Assume $(1/2)\lambda^u(\delta_Q) \neq \lambda_m^u$.

Lemma 3.1. *For any $\varepsilon > 0$ there exists $N > 0$ such that for all $n \geq N$,*

$$P(t_n) < h(\mu) - t_n\lambda^u(\mu) + \varepsilon.$$

Proof. Suppose the statement is false. Then, there exists $\varepsilon > 0$ such that $P(t_n) \geq h(\mu) - t_n\lambda^u(\mu) + \varepsilon$ holds for infinitely many n . For these n we have

$$P(t_n) = h(\mu_{t_n}) - t_n\lambda^u(\mu_{t_n}) \geq h(\mu) - t_n\lambda^u(\mu) + \varepsilon.$$

Since μ is Lyapunov minimizing by Theorem B, $\lambda^u(\mu) \leq \lambda^u(\mu_{t_n})$. Using this and the upper semi-continuity of entropy, for sufficiently large n we have

$$h(\mu) + \frac{\varepsilon}{2} - t_n\lambda^u(\mu) > h(\mu_{t_n}) - t_n\lambda^u(\mu_{t_n}).$$

These two inequalities yield a contradiction. \square

Suppose there exists a Lyapunov minimizing measure ν such that $h(\mu) < h(\nu)$. Let $\varepsilon > 0$ be such that $h(\mu) + \varepsilon < h(\nu)$. By Lemma 3.1, for sufficiently large n we have

$$P(t_n) < h(\mu) - t_n\lambda^u(\mu) + \varepsilon < h(\nu) - t_n\lambda^u(\mu) = h(\nu) - t_n\lambda^u(\nu),$$

a contradiction. \square

3.4. Convergence point of equilibrium measures as $t \rightarrow -\infty$. We now prove Theorem D.

Proof of Theorem D. The lemma below shows that δ_Q is the unique measure which maximizes the unstable Lyapunov exponent.

Lemma 3.2. *For any $\mu \in \mathcal{M}(f) \setminus \{\delta_Q\}$, $\lambda^u(\mu) < \lambda^u(\delta_Q)$.*

Proof. From the ergodic decomposition theorem, we only have to consider ergodic measures.

One can choose a neighborhood W of Q such that for any ergodic $\mu \in \mathcal{M}(f) \setminus \{\delta_Q\}$ with $\text{supp}(\mu) \cap I(\delta) = \emptyset$, $\text{supp}(\mu) \cap W = \emptyset$ holds. Hence, for such μ , $\lambda^u(\mu) < \lambda^u(\delta_Q)$ holds.

It is left to consider the case $\text{supp}(\mu) \cap I(\delta) \neq \emptyset$. Then $\mu(I(\delta)) > 0$, and from the Ergodic Theorem, it is possible to take a point $x \in \Omega$ such that

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} \log \|D_{f^i x} f|E_{f^i x}^u\| = \lambda^u(\mu),$$

and

$$\lim_{n \rightarrow \infty} \frac{1}{n} \#\{0 \leq i \leq n-1: f^i x \in I(\delta)\} = \mu(I(\delta)).$$

Let $0 \leq n_1 < n_2 < \dots$ be the sequence of integers such that $f^{n_k} x \in I(\delta)$ and $f^{n_k+i} x \notin I(\delta)$ for every $1 \leq i \leq n_{k+1} - n_k - 1$. Let p_k denote the bound period for $f^{n_k} x$. Then

$$\|D_{f^{n_k} x} f^{p_k}|E_{f^{n_k} x}^u\| \leq C e^{\frac{\lambda^u(\delta_Q)}{2} p_k}.$$

For iterates out of $I(\delta)$,

$$\|D_{f^{n_k+p_k}x} f^{n_{k+1}-n_k-p_k} |E_{f^{n_k+p_k}x}^u\| \leq C e^{\lambda^u(\delta_Q)(n_{k+1}-n_k-p_k)}.$$

Since $p_k \geq -C \log \delta$ by [18], $C e^{-\frac{\lambda^u(\delta_Q)}{2} p_k} \leq \delta^C$. Hence

$$\|D_{f^{n_k}x} f^{n_{k+1}-n_k} |E_{f^{n_k}x}^u\| \leq C e^{-\frac{\lambda^u(\delta_Q)}{2} p_k} \cdot e^{\lambda^u(\delta_Q)(n_{k+1}-n_k)} \leq \delta^C e^{\lambda^u(\delta_Q)(n_{k+1}-n_k)}.$$

This yields

$$\begin{aligned} \lim_{k \rightarrow \infty} \frac{1}{n_k} \sum_{i=0}^{n_k-1} \log \|D_{f^i x} f |E_{f^i x}^u\| &\leq \lambda^u(\delta_Q) + C \log \delta \cdot \lim_{k \rightarrow \infty} \frac{1}{n_k} \#\{0 \leq i \leq n_k - 1 : f^i x \in I(\delta)\} \\ &= \lambda^u(\delta_Q) + C \log \delta \cdot \mu(I(\delta)) < \lambda^u(\delta_Q). \end{aligned} \quad \square$$

Let $\{\mu_{t_n}\}_{n=0}^\infty$ be a sequence in $\mathcal{M}(f)$ such that $t_n \searrow -\infty$, μ_{t_n} is an ergodic equilibrium measure for φ_{t_n} and $\mu_{t_n} \rightarrow \mu$ as $n \rightarrow \infty$. We have $\sup\{\lambda^u(\mu) : \mu \in \mathcal{M}(f)\} = \limsup_{n \rightarrow \infty} \lambda^u(\mu_{t_n})$, and the upper semi-continuity of the unstable Lyapunov exponent in (6) gives $\limsup_{n \rightarrow \infty} \lambda^u(\mu_{t_n}) \leq \lambda^u(\mu)$. By Lemma 3.2, $\mu = \delta_Q$. Hence Theorem D holds. \square

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